EQUILIBRIUM OF AN ISOLATED CRACK IN AN ELASTIC STRIP

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INTRODUCTION

In [1-3], a solution was given to the problem of an infinite strip of width 2h containing a longitudinal symmetrically arranged crack of length $2l$. Three types of boundary conditions at the lateral surfaces of the strip were considered: contact with absolutely rigid, smooth bases (this condition is realized in the problem of a periodic system of parallel cracks of identical length), the boundaries of the strip free from loads, and conditions of rigid attachment (displacements at the lateral surfaces of the strip equal to zero). The solutions obtained are valid in general form for $h/l \gg 1$ (in [3], for $h/l \geqslant 2$). In [4, 5], the Wiener–Hopf method was used to obtain solutions of this problem in the limiting case where $h/l \ll 1$. The present article gives a general solution of the problem under the condition that the crack is disposed parallel to the boundaries of the strip, but not necessarily at an identical distance from them. The limiting equilibrium of a crack in a strip in this case will be determined by the two stress-concentration factors K_I and K_{II}, in distinction from the symmetrical case, where the limiting equilibrium is determined by only the one coefficient K_I. For different boundary conditions, the dependences of K_I and K_{II} are plotted as a function of the ratio of the distance between the crack and the nearest boundary to the half length of the crack.

§1. Under the assumption that the deformed state is symmetrical with respect to the axis x = 0, the general solution of the equations of equilibrium of an isotropic elastic body

$$(1-2\nu)\Delta\mathbf{u} + \operatorname{grad}\operatorname{div}\mathbf{u} = 0, \quad \mathbf{u} = u\cdot\mathbf{i} + w\cdot\mathbf{j}$$

can be written in the form

$$\begin{gathered} w(x, y) = \frac{2}{\pi}\int_0^\infty [(A + Bsy)\operatorname{sh}(sy) + (C + Dsy)\operatorname{ch}(sy)]\cos(sx)\,ds, \\ -u(x, y) = \frac{2}{\pi}\int_0^\infty [(A + Dsy + \gamma B)\operatorname{sh}(sy) + (A + Bsy + \gamma D)\operatorname{ch}(sy)]\sin(sx)\,ds, \end{gathered} \tag{1.1}$$

where $\gamma = (3-4)\nu$; A, B, C, and D are arbitrary functions of s. The connection between the components of the stress and deformation tensors is given by Hooke's law,

$$\begin{gathered} \sigma_{yy} = 2(1-2\nu)^{-1}[(1-\nu)\partial u/\partial x + \nu\partial w/\partial y], \\ \sigma_{xx} = 2(1-2\nu)^{-1}[\nu\partial u/\partial x + (1-\nu)\partial w/\partial y], \\ \sigma_{xy} = \partial u/\partial y + \partial w/\partial x. \end{gathered} \tag{1.2}$$

The formulas (1.1) and (1.2) contain dimensionless quantities. For simplicity in writing, the primes are omitted

$$\langle x, y, u, w, h\rangle' = \frac{\langle x, y, u, w, h\rangle}{l}, \quad \sigma'_{ij} = \sigma_{ij}/\mu, \quad \mu = E/2(1+\nu).$$

We consider an elastic layer $-h_2 \leqslant y \leqslant h_1$, $|x| < \infty$ having a cut, arranged with y = 0 and $|x| \leqslant 1$, under conditions of plane deformation. At the lateral surfaces of the layer with $y = h_1$ and $y = -h_2$, $x < \infty$, the satisfaction of the following boundary conditions is assumed:

The layer is located between absolutely rigid smooth slabs,

$$\sigma_{xy} = 0, \ w = 0; \tag{1.3}$$

the boundaries of the layer are free,

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$$\sigma_{yy} = 0,\ \sigma_{xy} = 0; \tag{1.4}$$

the bases of the layer are rigidly fixed,

$$u = 0,\ w = 0. \tag{1.5}$$

At the surface of the cut with $y = 0$ and $|x| \leqslant 1$,

$$\sigma_{yy} = -p(x)/\mu,\ \sigma_{xy} = -\tau(x)/\mu. \tag{1.6}$$

In the layer $-h_2 \leqslant y \leqslant h_1$ we separate out two regions: the first region $0 \leqslant y \leqslant h_1$, $|x| < \infty$ and the second $-h_2 \leqslant y \leq 0$, $|x| < \infty$. Values related to the first region will be denoted by the subscript 1, while those related to the second region will be denoted by the subscript 2. The form of the solution (1.1) will be common for these regions. Thus, to determine the eight unknown functions $A_i(s)$, $B_i(s)$, $C_i(s)$, and $D_i(s)$ $(i = 1, 2)$ we have six boundary conditions: four at the lateral surface of the strip and two at the surface of the cut. In addition to the boundary conditions, we also must satisfy the conditions of the continuity of the stresses for $y = 0$ and the continuity of the displacements for $y = 0$ and $|x| > 1$:

$$\begin{aligned} &\left.\begin{aligned}\sigma_{1yy} - \sigma_{2yy} = 0,\\ \sigma_{1xy} - \sigma_{2xy} = 0\end{aligned}\right\} \text{ for } |x| < \infty,\\ &\left.\begin{aligned}w_1 - w_2 = 0,\\ u_1 - u_2 = 0\end{aligned}\right\} \text{ for } |x| > 1. \end{aligned} \tag{1.7}$$

We solve the problem with the boundary conditions (1.3). In the case of the use of the boundary conditions (1.4) and (1.5), the solution is carried out completely analogously. Using (1.3) and the first pair of conditions from (1.7), we obtain

$$\begin{gathered} A_1 = \psi^{-1}[B_1\varphi_2 - B_2\varphi_1],\ A_2 = \psi^{-1}\left[B_1\varphi_1^1 - B_2\varphi_2^1\right],\\ C_1 = \psi^{-1}[B_1\varphi_4 + B_2\varphi_3],\ C_2 = \psi^{-1}\left[B_1\varphi_3^1 + B_2\varphi_4^1\right],\\ D_1 = -B_1 \operatorname{cth} H_1,\ D_2 = B_2 \operatorname{cth} H_2, \end{gathered}$$

where $H_1 = s h_1$; $H_2 = s h_2$; $\varphi_1 = \varphi_1(H_1, H_2) = H_2 \cosh H_1 \cdot \sinh^{-1} H_2 + \cosh H_1 \cdot \cosh H_2$; $\varphi_2 = \varphi_2(H_1, H_2) = H_1 \cosh H_2 \cdot \sinh^{-1} H_1 + 2(1-\nu)\cosh H_1 \cdot \cosh H_2 + (1-2\nu)\cosh^2 H_1 \cdot \sinh H_2 \cdot \sinh^{-1} H_1$; $\varphi_3 = \varphi_3(H_1, H_2) = H_2 \sinh \cdot H_1 \sinh^{-1} H_2 + \cosh H_2 \cdot \sinh H_1$; $\varphi_4 = \varphi_4(H_1, H_2) = H_1 \sinh \cdot H_2 \sinh^{-1} \cdot H_1 - 2(1-\nu)\cosh H_2 \cdot \sinh H_1 (1-2\nu)\cosh H_1 \cdot \sinh H_2$; $\psi = \sinh(H_1 + H_2)$, and the functions φ_i^1 $(i = 1, 2, 3, 4)$ are constructed according to the rule

$$\varphi_i^1 = \varphi_i(H_2, H_1).$$

The remaining mixed boundary conditions (1.6) and the second pair of conditions from (1.7) can be written using only the functions B_1 and B_2 in the form of pairwise integral equations:

$$\begin{aligned} &\left.\begin{aligned} &\frac{4}{\pi}\int_0^\infty s\left[B_1\varphi_1^1 - B_2\varphi_1\right]\psi^{-1}\cos(sx)\,ds = -p(x)/\mu,\\ &\frac{4}{\pi}\int_0^\infty s\left[B_1\varphi_3^1 + B_2\varphi_3\right]\psi^{-1}\sin(sx)\,ds = \tau(x)/\mu \end{aligned}\right\} x < 1,\\ &\left.\begin{aligned} &4(1-\nu)\pi^{-1}\int_0^\infty [B_1 - B_2]\cos(sx)\,ds = 0,\\ &4(1-\nu)\pi^{-1}\int_0^\infty s[B_1 \operatorname{cth} H_1 + B_2 \operatorname{cth} H_2]\cos(sx)\,ds = 0 \end{aligned}\right\} x > 1. \end{aligned} \tag{1.8}$$

The last equation corresponds to the condition $\partial u_1/\partial x - \partial u_2/\partial x = 0$ for $y = 0$ and $|x| > 1$.

§2. It is obvious that the last two equations of system (1.8) can be rewritten in the form

$$B_1(s) - B_2(s) = \int_0^1 w_0(x)\cos(sx)\,dx, \tag{2.1}$$

$$s[B_1(s)\,\mathrm{cth}\,H_1+B_2(s)\,\mathrm{cth}\,H_2]=\int_0^1 u_0(x)\cos(sx)\,dx,$$

where the functions $w_0(x)$ and $u_0(x)$ are proportional, respectively, to the functions $w_1(x, 0) - w_2(x, 0)$ and $\partial u_1/\partial x - \partial u_2/\partial x$ for $y = 0$.

Using the asymptotic behavior of the displacements in the neighborhood of the tip of the crack, we can represent $w_0(x)$ and $u_0(x)$ in the form of integrals of some functions $\varphi(t)$ and $\psi(t)$, continuous in the segment [0, 1] [6],

$$w_0(x)=\int_x^1 \frac{\tau\varphi(\tau)\,d\tau}{\sqrt{\tau^2-x^2}},\quad u_0(x)=\frac{\delta}{\sqrt{1-x^2}}+\int_x^1 \frac{\psi(\tau)\,d\tau}{\sqrt{\tau^2-x^2}},\quad \delta=-\int_0^1 \psi(\tau)\,d\tau, \tag{2.2}$$

where the constant δ is determined from the condition $\lim\limits_{x\to 1}\int_0^x u_0(\xi)d\xi=0$; the latter assures satisfaction of the equality $u_1(x, 0) = u_2(x, 0)$ for $x \geqslant 1$.

Substituting (2.2) into (2.1), we obtain

$$\begin{gathered} B_1(s)-B_2(s)=\frac{\pi}{2}\int_0^1 \tau\varphi(\tau)J_0(s\tau)\,d\tau=\frac{\pi}{2}\Phi_0(s), \\ s\,[B_1(s)\,\mathrm{cth}\,H_1+B_2(s)\,\mathrm{cth}\,H_2]=\frac{\pi}{2}\left[\delta J_0(s)+\int_0^1 \psi(\tau)J_0(s\tau)\,d\tau\right]=\frac{\pi}{2}\Psi_0(s). \end{gathered} \tag{2.3}$$

From this we have

$$\begin{gathered} B_1(s)=\frac{\pi}{2}F_0^{-1}[\Psi_0(s)s^{-1}+\Phi_0(s)\,\mathrm{cth}\,H_2], \\ B_2(s)=\frac{\pi}{2}F_0^{-1}[\Psi_0(s)s^{-1}-\Phi_0(s)\,\mathrm{cth}\,H_1],\quad F_0=\mathrm{cth}\,H_1+\mathrm{cth}\,H_2. \end{gathered}$$

We substitute these expressions for $B_1(s)$ and $B_2(s)$ into the unused first two equations of system (1.8) and for $x < 1$ we obtain

$$\begin{gathered} \int_0^\infty [\Phi_0(s)f_2(s)+\Psi_0(s)f_1(s)]\sin(sx)\,ds=-\int_0^x p(x)/\mu\cdot dx, \\ \int_0^\infty [\Phi_0(s)f_4(s)+\Psi_0(s)f_3(s)]\,s\sin(sx)\,ds=\tau(x)/\mu, \end{gathered} \tag{2.4}$$

where the first equation from (1.8) is integrated with respect to x within the limits 0 to x, and

$$\begin{gathered} f_1(s)=2(s\psi F_0)^{-1}\left[\varphi_1^1-\varphi_1\right], \\ f_2(s)=2(\psi F_0)^{-1}\left[\varphi_1^1\,\mathrm{cth}\,H_2+\varphi_1\,\mathrm{cth}\,H_1\right], \\ f_3(s)=2(s\psi F_0)^{-1}\left[\varphi_3^1+\varphi_3\right],\quad f_4(s)=sf_1(s). \end{gathered}$$

For large values of s the function $f_1(s) \sim O(e^{-H_1}, e^{-H_2})$. Inplace of the functions $f_2(s)$ and $f_3(s)$, we introduce $g_2(s)$ and $g_3(s)$ using the equalities

$$\begin{gathered} f_2(s)=1+g_2(s),\quad g_2(s)\sim O(H_1e^{-H_1},\,H_2e^{-H_2}), \\ sf_3(s)=1+g_3(s),\quad g_3(s)\sim O(H_1e^{-H_2},\,H_2e^{-H_1}). \end{gathered}$$

Into the integral equations determined in this manner, we substitute the functions $f_2(s)$ and $f_3(s)$ and expressions (2.3) for $\Phi_0(s)$ and $\Psi_0(s)$. Changing the order of the integration and using the known integral

$$\int_0^\infty J_0(s\tau)\sin(sx)\,ds = (x^2-\tau^2)^{-1/2} \text{ with } x>\tau,$$

we obtain two Abel equations,

$$\int_0^x \frac{\tau\varphi(\tau)\,d\tau}{\sqrt{x^2-\tau^2}} = H(x),\quad \int_0^x \frac{\psi(\tau)\,d\tau}{\sqrt{x^2-\tau^2}} = H_0(x),$$

where

$$H(x) = -\mu^{-1}\int_0^x p(x)\,dx - \int_0^1 \tau\varphi(\tau)\,d\tau\int_0^\infty g_2(s)\;J_0(s\tau)\;\sin\;(sx)\,ds - \int_0^1 \psi(\tau)\,d\tau\int_0^\infty f_1(s)\,[J_0(s\tau)-J_0(s)]\sin(sx)\,ds;$$

$$H_0(x) = \tau(x)/\mu - \int_0^1 \psi(\tau)\,d\tau\int_0^\infty g_3(s)\,[J_0(s\tau)-J_0(s)]\sin(sx)\,ds - \int_0^1 \tau\varphi(\tau)\,d\tau\int_0^\infty sf_4(s)\,J_0(s\tau)\sin(sx)\,ds,$$

whose solutions have the form

$$\varphi(t) = \frac{2}{\pi}\int_0^t \frac{H'(x)\,dx}{\sqrt{t^2-x^2}},\quad \frac{\psi(t)}{t} = \frac{2}{\pi}\int_0^t \frac{H_0'(x)\,dx}{\sqrt{t^2-x^2}}.$$

Without loss of the generality in the discussion, in what follows we assume $\tau(x) = 0$ and $p(x) = p_0 = \text{const}$. Differentiating $H(x)$ and $H_0(x)$ with respect to x and carrying out calculations, we obtain a system of integral Fredholm equations of the second kind:

$$\begin{gathered}\varphi_0(t) + \int_0^1 \varphi_0(\tau)\,K_1(\tau,t)\,d\tau + \int_0^1 \psi_0(\tau)\,K_2(\tau,t)\,d\tau = -\sqrt{t},\\ \psi_0(t) + \int_0^1 \psi_0(\tau)\,K_3(\tau,t)\,d\tau + \int_0^1 \varphi_0(\tau)\,K_4(\tau,t)\,d\tau = 0,\ 0\leqslant t,\ \tau\leqslant 1,\end{gathered} \tag{2.5}$$

where

$$\varphi_0(t) = \varphi(t)\,t^{1/2}p_0^{-1}\mu;\ \psi_0(t) = \psi(t)\,t^{-1/2}p_0^{-1}\mu;$$

$$K_1(\tau,t) = \sqrt{\tau t}\int_0^\infty sg_2(s)\,J_0(st)\,J_0(s\tau)\,ds;$$

$$K_2(\tau,t) = \sqrt{\tau t}\int_0^\infty sf_1(s)[J_0(s\tau)-J_0(s)]\,J_0(st)\,ds;$$

$$K_3(\tau,t) = \sqrt{\tau t}\int_0^\infty sg_3(s)\,[J_0(s\tau)-J_0(s)]\,J_0(st)\,ds;$$

$$K_4(\tau,t) = \sqrt{\tau t}\int_0^\infty s^2 f_4(s)\,J_0(s\tau)\,J_0(st)\,ds.$$

With the use of the boundary conditions (1.4)(the boundaries of the layer are free), we obtain the same system of integral Fredholm equations (2.5), in which

$$\begin{gathered}-g_2(s) = 2(\psi F_0)^{-1}\{\mathrm{sh}^2 H_1\,\mathrm{sh}^2 H_2\,\mathrm{sh}^2(H_1+H_2) + H_1^2H_2^2\times\\ \times(\mathrm{sh}^2 H_1 + \mathrm{sh}^2 H_2) - H_1^3\,\mathrm{sh}^3 H_2\,\mathrm{ch}\,H_2 - H_2^3\,\mathrm{sh}^3 H_1\,\mathrm{ch}\,H_1 -\\ - H_2^2\,\mathrm{sh}^2 H_1\,(\mathrm{sh}^2 H_1 + \mathrm{ch}^2 H_1\,\mathrm{sh}^2 H_2 + \mathrm{sh}\,H_1\,\mathrm{sh}\,H_2\,\mathrm{ch}\,H_1\,\mathrm{ch}\,H_2) -\\ - H_1^2\,\mathrm{sh}^2 H_2\,(\mathrm{sh}^2 H_2 + \mathrm{ch}^2 H_2\,\mathrm{sh}^2 H_1 + \mathrm{sh}\,H_1\,\mathrm{sh}\,H_2\,\mathrm{ch}\,H_1\,\mathrm{ch}\,H_2) +\\ + H_1H_2\,(H_1^2\,\mathrm{sh}^2 H_2 + H_2^2\,\mathrm{sh}^2 H_1 + H_1\,\mathrm{sh}^2 H_2\,\mathrm{sh}\,H_1\,\mathrm{ch}\,H_1 +\\ + H_2\,\mathrm{sh}^2 H_1\,\mathrm{sh}\,H_2\,\mathrm{ch}\,H_2 - 2\,\mathrm{sh}^2 H_1\,\mathrm{sh}^2 H_2)\} + 1,\end{gathered} \tag{2.6}$$

$$f_1(s) = 2(sF_0)^{-1}\left[H_2^2 \operatorname{sh}^2 H_1 - H_1^2 \operatorname{sh}^2 H_2\right],$$
$$g_3(s) = 2F_0^{-1}\left[H_1H_2(H_1+H_2) - H_2 \operatorname{sh}^2 H_1 - H_1 \operatorname{sh}^2 H_2 + \right.$$
$$\left. + H_1^2 \operatorname{sh} H_2 \operatorname{ch} H_2 + H_2^2 \operatorname{sh} H_1 \operatorname{ch} H_1 - \operatorname{sh} H_1 \operatorname{sh} H_2 \operatorname{sh}(H_1+H_2)\right] - 1,$$
$$f_4(s) = sf_1(s), \quad F_0 = [(H_1+H_2)^2 - \operatorname{sh}^2(H_1+H_2)],$$
$$\psi = H_1 \operatorname{sh}^2 H_2 + H_2 \operatorname{sh}^2 H_1 + \operatorname{sh} H_1 \operatorname{sh} H_2 \operatorname{sh}(H_1+H_2).$$

§3. The condition of the limiting equilibrium of the crack is completely determined by the stress-concentration factors K_I and K_{II} at the tip of the crack with a singularity of the order $(\Delta x)^{-1/2}$ ($\Delta x \ll 1$). We shall show that the components of the stress tensor have a singularity of the first order, and we shall find the coefficients for this singularity. Using the solution obtained — (1.1) and (2.3) — we can write

$$\sigma_{xx}(x,0) = -2\int_0^\infty [\Psi_0(s) f_5(s) + \Phi_0(s) f_6(s)]\, s \cos(sx)\, ds,$$
$$f_5(s) = (s\psi F_0)^{-1}[H_1 \operatorname{ch} H_2 \operatorname{sh}^{-1} H_1 - H_2 \operatorname{ch} H_1 \operatorname{sh}^{-1} H_2 -$$
$$- 2 \operatorname{ch} H_1 \operatorname{ch} H_2 - 2\operatorname{ch}^2 H_1 \frac{\operatorname{sh} H_2}{\operatorname{sh} H_1}\Big],$$
$$f_6(s) = (\psi F_0)^{-1}[H_1 \operatorname{ch} H_2 \operatorname{sh}^{-1} H_1 \operatorname{cth} H_2 + H_2 \operatorname{ch} H_1 \operatorname{sh}^{-1} H_2 \operatorname{cth} H_1 - F_0 \operatorname{ch} H_1 \operatorname{ch} H_2].$$

In this expression and in expressions (2.4) for σ_{yy} and σ_{xy}, separating out the principal terms in the functions $f_i(s)$ ($i = 1, 2, \ldots, 6$), we write

$$\sigma_{xx}(x,0) \simeq \int_0^\infty [2\Psi_0(s) + s\Phi_0(s)] \cos(sx)\, ds, \tag{3.1}$$
$$\sigma_{yy}(x,0) \simeq \int_0^\infty s\Phi_0(s)\cos(sx)\, ds, \quad \sigma_{xy}(x,0) \simeq -\int_0^\infty \Psi_0(s)\sin(sx)\, ds.$$

In the expressions for $\Phi_0(s)$ and $\Psi_0(s)$ we also take only the main integral part, since it can be shown that the remaining terms do not participate in the formation of the singularity:

$$\Phi_0(s) = \int_0^1 \tau\varphi(\tau) J_0(s\tau)\, d\tau = s^{-1}\left[\varphi(1) J_1(s) - \int_0^1 \varphi'(\tau)\,\tau J_1(s\tau)\, d\tau\right] \simeq$$
$$\simeq s^{-1}\varphi(1) J_1(s) - \ldots, \quad \Psi_0(s) \simeq \delta J_0(s) + \ldots$$

Substituting these relationships into (3.1), we obtain

$$\sigma_{xx}(x,0) \simeq \varphi(1)\int_0^\infty J_1(s)\cos(sx)\, ds + 2\delta\int_0^\infty J_0(s)\cos(sx)\, ds,$$
$$\sigma_{yy}(x,0) \simeq \varphi(1)\int_0^\infty J_1(s)\cos(sx)\, ds, \quad \sigma_{xy}(x,0) \simeq -\delta\int_0^\infty J_0(s)\sin(sx)\, ds.$$

Now, using known integrals [7], we can write an asymptotic representation of the components of the stress tensor ($\varepsilon \ll 1$) at the point $x = 1$:

$$\sigma_{xx}(x,0) \simeq -\varphi(1)[x^2-1]^{-1/2}(x+\sqrt{x^2-1})^{-1},$$
$$\sigma_{yy}(x,0) \simeq -\varphi(1)[x^2-1]^{-1/2}(x+\sqrt{x^2-1})^{-1},$$
$$\sigma_{xy}(x,0) \simeq -\delta[x^2-1]^{-1/2} \text{ for } x = 1+\varepsilon;$$
$$\sigma_{xx}(x,0) \simeq 2\delta[1-x^2]^{-1/2} + \varphi(1),$$
$$\sigma_{yy}(x,0) \simeq \varphi(1), \; \sigma_{xy}(x,0) \simeq 0$$

for $x = 1 - \varepsilon$. From this, it can be seen that the expressions have the required singularity in the neighborhood of the point $x = 1$. The coefficients with this singularity, i.e., the stress-concentration factors, are expressed in terms of the solutions of the integral Fredholm equations (2.5):

$$K_I = -p_0\varphi_0(1)\sqrt{l/2},\quad K_{II} = p_0\sqrt{l/2}\cdot\int_0^1 \psi_0(t)\sqrt{t}\,dt.$$

Let us examine two limiting cases in more detail. The first, where the crack is located symmetrically in the layer, i.e., $h_1 = h_2$, and the second where the crack is located near the boundary of a half space, i.e., $h_2 \to \infty$. For $h_1 = h_2 = h$, $K_2(\tau, t) = K_1(\tau, t) = 0$; consequently, $\psi(t) = 0$ and the problem is reduced to the solution of one integral equation for the function $\varphi_0(t)$:

$$\varphi_0(t) + \int_0^1 \varphi_0(\tau) K_{i0}(\tau, t)\,d\tau = -\sqrt{t},$$

$$K_{10}(\tau, t) = \sqrt{\tau t}\int_0^\infty s\left[H\,\mathrm{sh}^{-2}H + \mathrm{cth}\,H - 1\right] J_0(st) J_0(s\tau)\,ds,$$

$$K_{20}(\tau, t) = -\sqrt{\tau t}\int_0^\infty s\left[\frac{H^2 - \mathrm{sh}^2 H}{H + \mathrm{sh}\,H\,\mathrm{ch}\,H} + 1\right] J_0(st) J_0(s\tau)\,ds, \tag{3.2}$$

$$K_{30}(\tau,t) = \sqrt{\tau t}\int_0^\infty s\left[\frac{H^2 + (1-2\nu)^2 + \gamma\,\mathrm{ch}^2 H}{\gamma\,\mathrm{ch}\,H\,\mathrm{sh}\,H - H} - 1\right] J_0(st) J_0(s\tau)\,ds,$$

where the first kernel corresponds to the boundary conditions (1.3), the second to the conditions (1.4), and the third to the conditions (1.5). The results of a numerical calculation of Eq. (3.2) with the kernels $K_{i0}(\tau, t)$ ($i = 1, 2, 3$) are shown in Figs. 1 and 2 by the curves 1 and in Fig. 3 by the curves 1-4, plotted for the Poisson coefficient $\nu = 0.15, 0.25, 0.35$, and 0.45, respectively. Curves 1 in Figs. 1 and 2 coincide completely with curves plotted in accordance with dependences given in [3]. The curves in Fig. 3 depend to a considerable degree on the Poisson coefficient ν. With an increase in ν, the curves converge more rapidly to the asymptotic formula [5]

$$K_I = p_0\sqrt{1-2\nu}\ \ (1-\nu)^{-1}\sqrt{h/2\pi}, \tag{3.3}$$

valid for $h \ll 1$. Thus, for $\nu = 0.45$, a divergence from values calculated using formula (3.3) starts with a ratio $h/l = 0.5$, while the asymptotic formulas [4, 5]

$$K_I = p_0\sqrt{h/2\pi},$$

$$K_I = p_0\sqrt{6h/\pi}\ [0.1267 + 0.6733\lambda^{-1} + 0.5\lambda^{-2} + (0.0104 - 0.1267\lambda^{-1} - 0.3367\lambda^{-2} - 1/6\cdot\lambda^{-3})/(0.6733 + \lambda^{-1})],\ \lambda = h/l,$$

corresponding to the boundary conditions (1.3) and (1.4) hold up to a ratio $h/l < 2$ with an accuracy of 1-2%. In [3-5], a complete solution is given to the symmetrical problem of a crack in a strip, with the exception of the case where the boundaries of the strip are rigidly fixed, i.e., the boundary conditions (1.5) are satisfied. In this case the solution for $0.5 < h/l < 2$ is practically indeterminate.

Where $h_2 \to \infty$, the crack is located near the boundary of the half space. In distinction from the symmetrical case, here neither of the stress-concentration factors K_I and K_{II} is equal to zero, and the total system of Fredholm equations (2.5), (2.6) must be solved with the boundary conditions (1.3) and (1.4). If the boundray of the half space is rigidly fixed (for $y = h$, $u = w = 0$), we arrive at a system analogous to (2.5):

$$\varphi_0(t) - \int_0^1 \varphi_0(\tau) K_1(\tau, t)\,d\tau - \int_0^1 \psi_0(\tau) K_2(\tau, t)\,d\tau = \sqrt{t},$$

$$\psi_0(t) - \int_0^1 \psi_0(\tau) K_3(\tau, t)\,d\tau - \int_0^1 \varphi_0(\tau) K_4(\tau, t)\,d\tau = 0, \tag{3.4}$$

$$K_1(\tau, t) = \sqrt{\tau t}\int_0^\infty s\left[2\frac{f_1 f_5 + f_4 f_3}{\psi F_0} + 1\right] J_0(s\tau) J_0(st)\,ds,$$

$$K_2(\tau, t) = \sqrt{\tau t}\int_0^\infty 2\frac{f_3 - f_1}{F_0} J_0(st)\left[J_0(s\tau) - J_0(s)\right] ds,$$

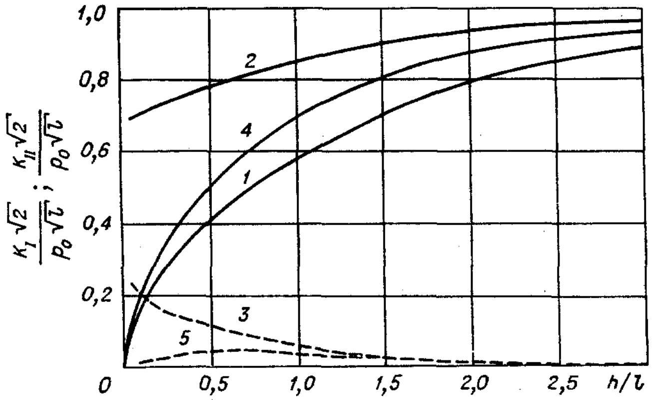

Fig. 1

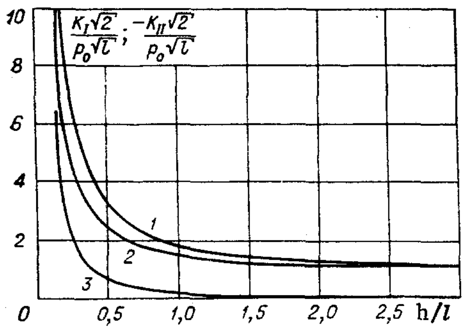

Fig. 2

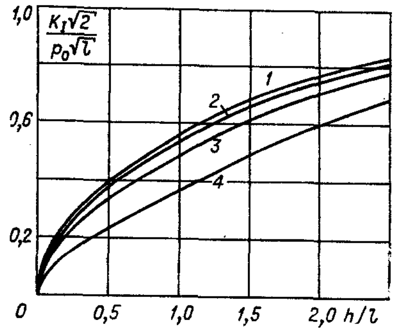

Fig. 3

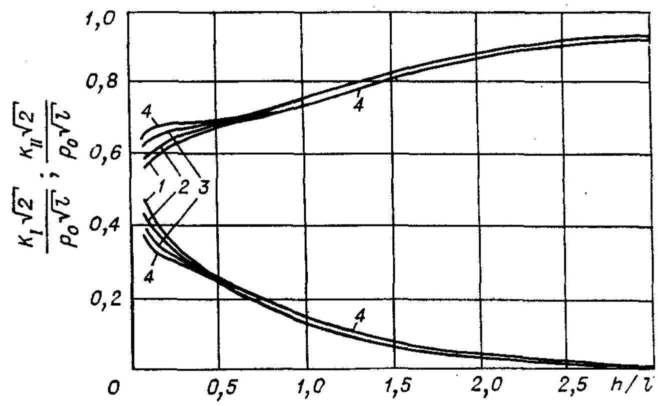

Fig. 4

$$K_3(\tau, t) = \sqrt{\tau t} \int_0^\infty s\left[2\frac{f_2 - f_1}{F_0} + 1\right] J_0(st)\,[J_0(s\tau) - J_0(s)]\,ds,$$

$$K_4(\tau, t) = \sqrt{\tau t} \int_0^\infty 2s^2 \frac{f_5 f_1 + f_4 f_2}{\psi F_0} J_0(s\tau)\, J_0(st)\, ds,$$

where

$$f_1 = H^2 + \gamma\,\mathrm{ch}^2 H + (1 - 2\nu)^2;\; F_0 = \gamma e^{2H};$$
$$f_2 = H - \gamma\,\mathrm{ch}\,H\,\mathrm{sh}\,H;\; \psi = H + (1 - 2\nu) - \gamma e^H\,\mathrm{ch}\,H;$$
$$f_3 = \gamma\,\mathrm{ch}^2 H - (1 - 2\nu);\; f_4 = H + 2(1 - \nu) + \gamma e^H\,\mathrm{sh}\,H;$$
$$f_5 = -H - 2(1 - \nu) + \gamma e^H\,\mathrm{ch}\,H.$$

Equations (2.5), (2.6) (as $h \to \infty$), and (3.4) were calculated numerically. The results of the calculations are given in Figs. 1, 2, and 4. In Figs. 1 and 2, curves 2 and 3 relate, respectively, to $K_I\sqrt{2}/p_0\sqrt{l}$ and $K_{II}\sqrt{2}/p_0\sqrt{l}$. Curves 1-4 in Fig. 4 represent $K_I\sqrt{2}/p_0\sqrt{l}$ (upper) and $K_{II}\sqrt{2}/p_0\sqrt{l}$ (lower), calculated for $\nu = 0.15, 0.25, 0.35$, and 0.45. As can be seen from the dependences given in Fig. 4, the effect of the Poisson coefficient ν starts to appear with $h/l \leqslant 0.5$, i.e., only with an approach of the crack to the boundary of the half space.

In Fig. 1, curves 4 and 5 illustrate $K_I\sqrt{2}/p_0\sqrt{l}$ and $K_{II}\sqrt{2}/p_0\sqrt{l}$, calculated using Eqs. (2.5) with $h_1 = h$ and $h_2 = 2h$, which corresponds to a crack located in a strip of width 3h, at a distance from the upper boundary equal to one third of its width. As can be seen from the curves given, for small values of h/l ($h/l < 1$), the value of the ratio $K_{II}\sqrt{2}/p_0\sqrt{l}$ varies only insignificantly, and the value of K_I is well described by the formula

$$K_I = \sqrt{3h/4\pi} \cdot p_0,$$

i.e., it behaves in the same way as in the case of a system of parallel cracks, arranged at an identical distance 3h apart.

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